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INVERSES OF TRIDIAGONAL TOEPLITZ AND PERIODIC MATRICES WITH APPLICATIONS TO MECHANICS†‡

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The linear algebraic equation $Ax = b$ with tridiagonal coefficient matrix A is solved by analytical matrix inversion. An explicit formula is known if A is a Toeplitz matrix. New formulas are presented for the following cases: (1) A is of Toeplitz type except that $A(1, 1)$ and $A(n, n)$ are different from the remaining diagonal elements. (2) A is p-periodic ($p > 1$), by which is meant that in each of the three bands of A a group of p elements is periodically repeated. (3) The tridiagonal matrix A is composed of periodic submatrices of different periods. In cases (2) and (3) the problem of matrix inversion is reduced to a second-order difference equation with periodic coefficients. The solution is based on Floquet's theorem. It is shown that for $p = 1$ the formulae found for periodic matrices reduce to special forms valid for Toeplitz matrices. The results are applied to problems of elastostatics and of vibration theory. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

The subject of this paper is systems of linear equations of the form

$$\beta_{i-1}x_{i-1}+\alpha_i x_i+\gamma_i x_{i+1}=b_i \quad (i=1,\ldots,\ n;\ x_0=x_{n+1}=0) \tag{1.1}$$

with real or complex coefficients. The matrix form is $Ax = b$ with a symmetric or asymmetric tridiagonal matrix A. Such systems of equations arise in many fields. In the simplest case the matrix elements are $\alpha_i \equiv \alpha$, $\beta_i \equiv \beta$, $\gamma_i \equiv \gamma$. Such matrices are called (tridiagonal) Toeplitz matrices. A Toeplitz matrix will be called perturbed if the "boundary" elements α_1 and α_n are different from α. These boundary elements reflect boundary conditions of the mechanical system under investigation. Mechanical systems which are composed of periodically repeated subsystems give rise to periodic matrices with elements $\alpha_{i+p} \equiv \alpha_i$, $\beta_{i+p} \equiv \beta_i$, $\gamma_{i+p} \equiv \gamma_i$ ($p > 1$).

In Section 2 the elements of the inverses of tridiagonal matrices are expressed in terms of continuants, i.e. of determinants of tridiagonal submatrices of A. Continuants are solutions of a linear second-order difference equation. The coefficients of this equation are the diagonal elements α_i and the products $\delta_i = \beta_i\gamma_i$ of the off-diagonal elements. From this it follows that the inversion of an asymmetric matrix is not more difficult than that of a symmetric matrix. Throughout this paper $\delta_i = \beta_i\gamma_i$ of the off-diagonal elements. From this it follows that the inversion of an asymmetric matrix is not more difficult than that of a symmetric matrix. Throughout this paper $\delta_i \neq 0$ ($i = 1, \ldots, n-1$) is assumed, since otherwise the problem of inversion can be reduced to the inversion of smaller tridiagonal matrices.

In Section 3 difference equations with constant coefficients $\alpha_i \equiv \alpha$, $\delta_i \equiv \delta$ are resolved explicitly. With these solutions explicit expressions are obtained for the inverse of Toeplitz matrices and of perturbed Toeplitz matrices. With the exception of some formulae related to perturbed matrices this materials is well known. New results are obtained in Sections 4 and 5. In Section 4 difference equations with p-periodic coefficients $\alpha_{i+p} \equiv \alpha_i$, $\delta_{i+p} \equiv \delta_i$ are investigated. Explicit solutions are based on Floquet's theorem. The result for p-periodic coefficients is identical with that for constant coefficients if p is set equal to 1. In Section 5 the results are further generalized to tridiagonal matrices with submatrices of different period lengths. In Section 6 the results are applied to elastostatics and to vibration problems.

Tridiagonal matrices and generalizations thereof have been studied in many papers [1–7]. In [5] inverses of periodic tridiagonal matrices are discussed without, however, giving closed-form expressions of the kind obtained in the present paper.

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2. THE INVERSE MATRIX EXPRESSED IN TERMS OF CONTINUANTS

Let u_i $(i = 1, \ldots, n)$ be the determinant of the tridiagonal submatrix with elements A_{jk} $(j, k = 1, \ldots, i)$ of the coefficient matrix A. Following the notation in [1] u_i is called the ith continuant of A. Furthermore, let υ_i $(i = 1, \ldots, n)$ be the ith continuant of the transpose of A about its secondary diagonal. From these definitions it follows that $u_n = \upsilon_n = \det A$. If A is persymmetric (symmetric to its secondary diagonal) then $\upsilon_i = u_i$ $(i = 1, \ldots, n)$.

The element $(A^{-1})_{ij}$ $(i, j = 1, \ldots, n)$ of the inverse of an arbitrary matrix A is C_{ij}/u_n where C_{ij} is the cofactor of A_{ji}. It is $(-1)^{i-j}$ times the determinant of the submatrix of A which remains after deleting row j and column i. For the matrix in Eq. (1) and for $i > j$ the cofactor is

$$C_{ij} = (-1)^{i-j} \begin{array}{r|ccccccc|}
 & & & & & & i-1 & i+1 \\
j-1 & \|A_{j-1}\| & \gamma_{j-1} & & & & & \\
j+1 & & \beta_j & \alpha_{j+1} & \gamma_{j+1} & & & \\
 & & & \cdot & \cdot & \cdot & & \\
 & & & & \beta_{i-3} & \alpha_{i-2} & \gamma_{i-2} & \\
 & & & & & \beta_{i-2} & \alpha_{i-1} & \\
 & & & & & & \beta_{i-1} & \gamma_i \\
 & & & & & & & \|B_{n-i}\|
\end{array} \quad (i > j)$$

$[A_{j-1}]$ and $[B_{n-i}]$ are the matrices with elements A_{kl} $(k, l = 1, \ldots, j-1)$ and $(k, l = i+1, \ldots, n)$, respectively. Their determinants are the continuants u_{j-1} and υ_{n-i}, respectively. It is easily shown that C_{ij} remains unchanged if all elements α and γ outside of $[A_{j-1}]$ and $[B_{n-i}]$ are replaced by zero. First, this is shown for γ_{j-1} and hence also for γ_i and then for the remaining elements. In each case one shows that in the expression for the determinant the coefficient of the element under consideration is zero. It follows that

$$C_{ij} = (-1)^{i-j} u_{j-1} \upsilon_{n-i} \prod_{k=j}^{i-1} \beta_k \quad (i > j;\ j = 1, \ldots, n)$$

Let $u_0 = \upsilon_0 = 1$. Then this formula is also valid in the case when $i = j$ $(i = 1, \ldots, n)$. If $i < j$, then i and j as well as β and γ must be interchanged. With this formula the elements of A^{-1} are

$$(A^{-1})_{ij} = \begin{cases} (-1)^{i-j} \dfrac{u_{j-1}\upsilon_{n-i}}{u_n} \displaystyle\prod_{k=j}^{i-1} \beta_k & (i \geqslant j) \\ (-1)^{j-i} \dfrac{u_{i-1}\upsilon_{n-j}}{u_n} \displaystyle\prod_{k=i}^{j-1} \gamma_k & (i \leqslant j) \end{cases} \quad (j = 1, \ldots, n) \tag{2.1}$$

Thus, A^{-1} is known as soon as the continuants are known. Expansion of the determinant u_i by the ith column yields the second-order linear difference equation

$$u_i = \alpha_i u_{i-1} - \delta_{i-1} u_{i-2} \quad (i = 2, \ldots, n) \tag{2.2}$$

where $\delta_i = \beta_i \gamma_i$ $(i = 1, \ldots, n-1)$ and with the initial conditions

$$u_0 = 1, \quad u_1 = \alpha_1 \tag{2.3}$$

The continuants υ_i $(i = 0, \ldots, n)$ are obtained by applying the same equation to the transpose of A about the secondary diagonal. Throughout this paper $\delta_i \neq 0$ $(i = 1, \ldots, n-1)$ is assumed. Under this condition no two successive continuants u_k and u_{k+1} or υ_k and υ_{k+1} $(k \geqslant 0)$ are zero. Equation (3) is the same for symmetric and for asymmetric matrices.

3. TOEPLITZ MATRICES

In this section Eq. (2.2) for continuants is explicitly solved for tridiagonal matrices with perturbations. This is the case $\beta_i \equiv \beta$, $\gamma_i \equiv \gamma$ $(i = 1, \ldots, n-1)$, $\alpha_i \equiv \alpha$, $(i = 2, \ldots, n-1)$, α_1, $\alpha_n \neq \alpha$. From the solution u_i ($i = 0, \ldots, n$) of Eq. (2.2) the continuants υ_i are obtained by interchanging α_1 and α_n. With $\delta = \beta\gamma$ Eq. (2.2) takes the form

$$u_i = \begin{cases} \alpha u_{i-1} - \delta u_{i-2} & (i = 2, \ldots, n-1) \\ \alpha u_{n-1} - \delta u_{n-2} + (\alpha_n - \alpha) u_{n-1} & (i = n) \end{cases} \tag{3.1}$$

with initial conditions (2.3). In the range $i = 2, \ldots, n-1$ it has constant coefficients. In this range the equation has a solution of the form $u_i = Cq^i$ ($i = 0, \ldots, n-1$). Substitution into Eq. (3.1) yields the characteristic equation

$$q^2 - \alpha q + \delta = 0 \tag{3.2}$$

In the case of different roots $q_1 \neq q_2$ the general solution is $u_i = C_1 q_1^i + C_2 q_2^i$ ($i = 0, \ldots, n-1$). The coefficients C_1 and C_2, determined by the initial conditions (2.3), are found to be $C_1 = [q_1 + (\alpha_1 - \alpha)]/(q_1 - q_2)$ and $C_2 = -[q_2 + (\alpha_1 - \alpha)]/(q_1 - q_2)$. Hence

$$u_i = \frac{q_1^{i+1} - q_2^{i+1} + (\alpha_1 - \alpha)(q_1^i - q_2^i)}{q_1 - q_2} \quad (i = 0, \ldots, n-1) \tag{3.3}$$

Eq. (5) then yields for u_n the expression

$$u_n = \frac{q_1^{n-1}(q_1 + \alpha_1 - \alpha)(q_1 + \alpha_n - \alpha) - q_2^{n-1}(q_2 + \alpha_1 - \alpha)(q_2 + \alpha_n - \alpha)}{q_1 - q_2} \tag{3.4}$$

In the case when $q_1 = q_2 = q = \alpha/2$ the general solution is $u_i = (C_1 + iC_2)(\alpha/2)^i$ ($i = 0, \ldots, n-1$). The initial conditions (2.3) yield $C_1 = 1$, $C_2 = 2\alpha_1/\alpha - 1$. From this it follows that

$$u_i = (1+i)q^i + i(\alpha_1 - \alpha)q^{i-1} \quad (i = 0, \ldots, n-1) \tag{3.5}$$

$$u_n = [q^2 + n(q + \alpha_1 - \alpha)(q + \alpha_n - \alpha) - (\alpha_1 - \alpha)(\alpha_n - \alpha)]q^{n-2} \tag{3.6}$$

In the special case $\alpha_1 = \alpha_n = \alpha$ Eqs (3.3) to (3.6) have the forms

$$u_i = \upsilon_i = \begin{cases} \dfrac{q_1^{i+1} - q_2^{i+1}}{q_1 - q_2} & (q_1 \neq q_2) \\ (1+i)q^i & (q_1 = q_2 = q = \alpha/2) \end{cases} \quad (i = 0, \ldots, n) \tag{3.7}$$

These results are known from [2, 3]. In [1] the case $q_1 = q_2$ is not considered and the result for $q_1 \neq q_2$ is misprinted.

For real matrices the continuants (3.3)–(3.7) are real even if $q_{1,2}$ are complex conjugate. In what follows real expressions will be developed for asymmetric matrices in the special case $\alpha_1 = \alpha_n = \alpha$ and for symmetric matrices in special cases with $\alpha_1 = \alpha_n \neq \alpha$.

Real asymmetric Toeplitz matrices

The matrix is persymmetric, whence it follows that $u_i = \upsilon_i$ ($i = 0, \ldots, n$) and Eq. (2.1) takes the form

$$(A^{-1})_{ij} = \begin{cases} (-\beta)^{i-j} \dfrac{u_{j-1} u_{n-i}}{u_n} & (i \geqslant j) \\ (-\gamma)^{j-i} \dfrac{u_{i-1} u_{n-j}}{u_n} & (i \leqslant j) \end{cases} \quad (j = 1, \ldots, n) \tag{3.8}$$

Without loss of generality it will be assumed that the diagonal elements α of A are non-negative (the case $\alpha < 0$ is treated by inverting $-A$). In Eq. (3.7) three cases must be distinguished depending on whether Eq. (3.2) has two different real roots, a double root or complex conjugate roots.

1. *Different real roots.* From Eq. (3.2) with $\alpha > 0$ it follows that $q_1 > 0$, sign q_2 = sign δ and $|q_2| < q_1$. We define the real number

$$r = \frac{q_2}{q_1} = \frac{\alpha - \sqrt{\alpha^2 - 4\delta}}{\alpha + \sqrt{\alpha^2 - 4\delta}} \quad (0 < |r| < 1;\ \operatorname{sign} r = \operatorname{sign} \delta) \tag{3.9}$$

From Eq. (3.2) it follows that $q_1 q_2 = \delta$ and with this from Eq. (3.9) we have $q_1 = \sqrt{\delta / r}$. In these terms Eq. (3.7) is written in the form

$$u_i = \left(\frac{\delta}{r}\right)^{i/2} \frac{1 - r^{i+1}}{1 - r} \quad (i = 0, \ldots, n) \tag{3.10}$$

2. *Double root*: $q = \alpha/2 = \sqrt{\delta} > 0$. With this, Eq. (3.7) becomes

$$u_i = \delta^{i/2}(1 + i) \quad (i = 0, \ldots, n) \tag{3.11}$$

3. *Complex conjugate roots*. In this case $4\delta - \alpha^2 > 0$ and, thus, also $\delta > 0$. Since $\alpha \geqslant 0$ the real part is $\geqslant 0$. The absolute value is $\sqrt{\delta}$. By writing $q_{1,2} = \sqrt{e^{\pm i\varphi}}$ we define the angle

$$\varphi = \operatorname{arctg} \frac{\sqrt{4\delta - \alpha^2}}{\alpha} \quad \left(0 < \varphi \leqslant \frac{\pi}{2}\right) \tag{3.12}$$

Equation (3.7) has the form

$$u_j = \delta^{j/2} \frac{e^{i(j+1)\varphi} - e^{-i(j+1)\varphi}}{e^{i\varphi} - e^{-i\varphi}} = \delta^{j/2} \frac{\sin[(j+1)\varphi]}{\sin \varphi} \quad (j = 0, \ldots n) \tag{3.13}$$

Equations (3.10), (3.11) and (3.13) together with (3.8) yield the final result for the inverse of A:

$$(A^{-1})_{ij} = \begin{cases} (-\beta)^{i-j} \left(\dfrac{r}{\delta}\right)^{(i-j+1)/2} \dfrac{(1 - r^j)(1 - r^{n+1-i})}{(1 - r)(1 - r^{n+1})} & (\alpha^2 > 4\delta) \\ (-\beta)^{i-j} \left(\dfrac{1}{\delta}\right)^{(i-j+1)/2} \dfrac{j(n + 1 - i)}{n + 1} & (\alpha^2 = 4\delta) \\ (-\beta)^{i-j} \left(\dfrac{1}{\delta}\right)^{(i-j+1)/2} \dfrac{\sin(j\varphi) \sin[(n + 1 - i)\varphi]}{\sin \varphi \sin[(n + 1)\varphi]} & (\alpha^2 < 4\delta) \\ & (j = 1, \ldots, n;\ i \geq j;\ \alpha > 0) \end{cases} \tag{3.14}$$

The expressions for $i < j$ are obtained by interchanging i and j as well as β and γ. A symmetric matrix is characterized by $\delta = \beta^2$ and $0 < r < 1$.

Real symmetric Toeplitz matrices with special perturbations

In this case Eqs (3.3)–(3.6) apply. In some mechanical systems with a symmetric matrix the special case $\alpha_1 = \alpha_n = \alpha + \beta$ is of particular interest. In what follows the more general case

$$\alpha_1 = \alpha_n = \alpha + \sigma_1 \beta \quad (\sigma_1 = +1 \text{ or } -1) \tag{3.15}$$

is considered. The elements of the inverse matrix A^{-1} can be given forms similar to those in Eq. (3.14). Because of the symmetry and persymmetry Eq. (3.8) has the special form

$$(A^{-1})_{ij} = (A^{-1})_{ji} = (-\beta)^{i-j} \frac{u_{j-1} u_{n-i}}{u_n} \quad (j = 1, \ldots, n;\ i \geqslant j) \tag{3.16}$$

As in the previous section $\alpha \geqslant 0$ is assumed without loss of generality. We define $\sigma_\beta = \operatorname{sign} \beta = \sigma = \sigma_1 \sigma_\beta$. First, Eqs (3.3) and (3.4) for real or complex roots $q_1 \neq q_2$ are considered. Since $q_1 q_2 = \beta^2$ we have

$$\alpha_1 - \alpha = \alpha_n - \alpha = \sigma_1 \beta = \sigma |\beta| = \sigma \sqrt{q_1 q_2}$$

From this it follows that

$$u_i = \frac{q_1^{i+1} - q_2^{i+1} + \sigma\sqrt{q_1 q_2}(q_1^i - q_2^i)}{q_1 - q_2} = \frac{(q_1^{i+½} - \sigma q_2^{i+½})(\sqrt{q_1} + \sigma\sqrt{q_2})}{q_1 - q_2} =$$

$$= \frac{q_1^{i+½} - \sigma q_2^{i+½}}{\sqrt{q_1} - \sigma\sqrt{q_2}} \quad (i = 0, \dots, n-1)$$

$$u_n = \frac{q_1^{n-1}(q_1 + \sigma\sqrt{q_1 q_2})^2 - q_2^{n-1}(q_2 + \sigma\sqrt{q_1 q_2})^2}{q_1 - q_2} = \frac{\sqrt{q_1} + \sigma\sqrt{q_2}}{\sqrt{q_1} - \sigma\sqrt{q_2}}(q_1^n - q_2^n)$$

1. *Real roots $q_1 \neq q_2$ are positive*. Hence, the parameter r defined by Eq. (3.9) is in the range $0 < r < 1$. Furthermore, $q_1 = |\beta| r^{-1/2}$. This yields

$$u_i = \left(\frac{|\beta|}{\sqrt{r}}\right)^i \frac{1 - \sigma r^{i+½}}{1 - \sigma r^{½}} \quad (i = 0, \dots, n-1) \tag{3.17}$$

$$u_n = \left(\frac{|\beta|}{\sqrt{r}}\right)^i (1 - r^n) \frac{1 + \sigma r^{½}}{1 - \sigma r^{½}} \tag{3.18}$$

2. *Complex roots*. Again, Eq. (3.12) is applicable

$$q_{1,2} = |\beta| e^{\pm i\varphi}, \quad \varphi = \operatorname{arctg}\frac{\sqrt{4\beta^2 - \alpha^2}}{\alpha} \quad \left(0 < \varphi \leqslant \frac{\pi}{2}\right)$$

This yields

$$u_i = \begin{cases} |\beta|^i \dfrac{\sin[(i+½)\varphi]}{\sin(\varphi/2)} & (\sigma = +1) \\ |\beta|^i \dfrac{\cos[(i+½)\varphi]}{\cos(\varphi/2)} & (\sigma = -1) \end{cases} \quad (i = 0, \dots, n-1) \tag{3.19}$$

$$u_n = \begin{cases} 2|\beta|^n \operatorname{ctg}(\varphi/2) \sin n\varphi & (\sigma = +1) \\ -2|\beta|^n \operatorname{tg}(\varphi/2) \sin n\varphi & (\sigma = -1) \end{cases} \tag{3.20}$$

3. *A double root requires $\alpha/2 = |\beta|$*. We write $A = \alpha/2A^*$ and consider the matrix A^* first. It has the elements $\alpha_i^* = 2$, $\alpha_1^* = \alpha_n^* = 2 + \sigma$, $\beta^* = \sigma_\beta$ and the double root $q = 1$. Eqs (3.5) and (3.6) reduce to

$$u_i^* = 1 + i(1+\sigma) \quad (i = 0, \dots, n-1), \quad u_n^* = n(1+\sigma)^2 \tag{3.21}$$

With this the continuants of A are

$$u_i = |\beta|^i u_i^* (i = 1, \dots, n)$$

With Eqs (3.17)–(3.21), Eq. (3.16) yields the final result for the inverse matrix

$$(A^{-1})_{ij} = (A^{-1})_{ji} = \begin{cases} \dfrac{1}{|\beta|}(-\sigma_\beta)^{i-j} r^{(i-j+1)/2} \dfrac{(1 - \sigma r^{j-½})(1 - \sigma r^{n-i+½})}{(1-r)(1-r^n)} & & (\alpha^2 > 4\beta^2) \\ \dfrac{1}{|\beta|}(-\sigma_\beta)^{i-j} \dfrac{(2j-1)[2(n-i)+1]}{4n} & (\sigma = +1) & (\alpha^2 = 4\beta^2) \\ A \text{ is singular} & (\sigma = -1) & (\alpha^2 = 4\beta^2) \\ \dfrac{1}{|\beta|}(-\sigma_\beta)^{i-j} \dfrac{\sin[(j-½)\varphi]\sin[(n-i+½)\varphi]}{\sin\varphi \sin n\varphi} & (\sigma = +1) & (\alpha^2 < 4\beta^2) \\ \dfrac{1}{|\beta|}(-\sigma_\beta)^{i-j} \dfrac{\cos[(j-(½))\varphi]\cos[(n-i+(½))\varphi}{\sin\varphi \sin n\varphi} & (\sigma = -1) & (\alpha^2 < 4\beta^2) \end{cases} \tag{3.22}$$

$$(j = 1, \dots, n, \; i \geqslant j; \; \alpha > 0)$$

4. PERIODIC TRIDIAGONAL MATRICES

The inversion of a p-periodic tridiagonal matrix requires the solution of the linear difference equation (2.2) with p-periodic coefficients $\alpha_{i+p} \equiv \alpha_i$, $\delta_{i+p} \equiv \delta_i$. According to Floquet's theorem it has at least one solution of the form

$$u_i = \lambda_{i+1} q^i \quad (\lambda_{i+p} \equiv \lambda_i) \tag{4.1}$$

with p-periodic coefficients. Substitution into Eq. (2.2) results in the equations

$$-\frac{\delta_{i-1}}{q}\lambda_{i-1} + \alpha_i\lambda_i - q\lambda_{i+1} = 0 \quad (i = 1,\ldots,\ p;\ \delta_0 = \delta_p) \tag{4.2}$$

This represents an eigenvalue problem with eigenvalues q and eigenvector components $\lambda_0, \ldots, \lambda_{p-1}$. The special case $u_{p-1} = 0$ is solved first. From (2.2) it follows that $u_{p+1} = \alpha_1 u_p = u_1 u_p$, $u_{p+2} = \alpha_{p+2} u_p u_1 - \delta_{p+1} u_{p+1} u_p = (\alpha_2 u_1 - \delta_1) u_p = u_2 u_p$ etc. Using the principle of induction it is easily shown that the solution satisfying the prescribed initial conditions can be written in the form

$$u_{mp+i} = u_i u_p^m \quad (m = 0, 1, \ldots,;\ i = 0, \ldots, p-1) \tag{4.3}$$

This is Eq. (4.1) with $q^p = u_p$ and $\lambda_{i+1} = u_i/q^i$ $(i = 0, \ldots, p-1)$. Hence, u_p is one of the two roots of the quadratic equation. The other root does not appear because of the initial conditions. The continuants are independent of δ_p.

In the general case $u_{p-1} \neq 0$ with $\lambda_p \neq 0$ we are free to choose $\lambda_p = 1$. The case $p = 2$ with $\lambda_2 = 1$ must be considered separately. Equations (4.2), after eliminating λ_1 from the first equation, are then

$$q^4 - (\alpha_1\alpha_2 - \delta_1 - \delta_2)q^2 + \delta_1\delta_2 = 0, \quad \lambda_1 = (q^2 + \delta_2)/(q\alpha_1) \tag{4.4}$$

The first of these two equations is the characteristic quadratic equation (quadratic for q^2).

In the general case $p > 2$ Eqs (4.2) are written in the form

$$\begin{Vmatrix} \alpha_1 & -q & & & & \\ -\delta_1/q & \alpha_2 & -q & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & -\delta_{p-3}/q & \alpha_{p-2} & -q \\ & & & & -\delta_{p-2}/q & \alpha_{p-1} \end{Vmatrix} \begin{Vmatrix} \lambda_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \lambda_{p-1} \end{Vmatrix} = \begin{Vmatrix} \delta_p/q \\ 0 \\ \cdot \\ \cdot \\ 0 \\ q \end{Vmatrix} \tag{4.5}$$

$$-\frac{\delta_{p-1}}{q}\lambda_{p-1} + \alpha_p - q\lambda_1 = 0 \tag{4.6}$$

Let the asymmetric tridiagonal coefficient matrix of Eq. (4.5) be called $\hat{A}$. The matrices $\hat{A}$ and A have in common the diagonal elements α_i and the products $\delta_i = \beta_i\gamma_i$ $(i = 1, \ldots, p-1)$ of the off-diagonal elements and, hence, also the continuants u_i $(i = 0, \ldots, p-1)$. The determinant is $u_{p-1} \neq 0$. The continuants of the transpose of $\hat{A}$ about the secondary diagonal are called $\hat{\upsilon}_i$ $(i = 1, \ldots, p-1)$. In addition $\hat{\upsilon}_{-1} = 0$ and $\hat{\upsilon}_0 = 1$ are defined. With Eq. (2.1) the inverse of $\hat{A}$ has the elements

$$(\hat{A}^{-1})_{ij} = \begin{cases} q^{j-i}\dfrac{u_{j-1}\hat{\upsilon}_{p-i-1}}{u_{p-1}}\prod_{k=j}^{i-1}\delta_k & (i \geqslant j) \\ q^{j-i}\dfrac{u_{i-1}\hat{\upsilon}_{p-j-1}}{u_{p-1}} & (i \leqslant j) \end{cases} \quad (j = 1, \ldots, p-1) \tag{4.7}$$

With this Eq. (4.5) yields

$$\lambda_i = \frac{\delta_p}{q}(\hat{A}^{-1})_{i1} + q(\hat{A}^{-1})_{i,p-1} = f_i(q^p)\frac{q^{-i}}{u_{p-1}} \quad (i = 1, \ldots, p) \tag{4.8}$$

with the abbreviation

$$f_i(q^p) = u_{i-1}q^p + \hat{v}_{p-i-1}\prod_{j=0}^{i-1}\delta_j \quad (i = 1, \ldots, p) \tag{4.9}$$

This formula is valid also for $i = p$ in which case it yields $\lambda_p = 1$. The eigenvalue q itself is found from Eq. (4.6). Substitution of the expressions for λ_1 and λ_{p-1} yields, after simple rearrangements, the characteristic quadratic equation (quadratic for q^p)

$$q^{2p} - (u_p - \delta_p\hat{v}_{p-2})q^p + \prod_{j=1}^{p}\delta_j = 0 \tag{4.10}$$

In deriving Eqs (4.8)–(4.10) $p > 2$ was assumed. However, in the case $p = 2$ they are identical with Eqs (4.4) so that they apply to arbitrary $p > 1$. The coefficient of q^p is

$$u_p - \delta_p\hat{v}_{p-2} = \sum(-1)^{k_1+\ldots+k_p}\alpha_1^{j_1}\alpha_2^{j_2}\ldots\alpha_p^{j_p}\delta_1^{k_1}\delta_2^{k_2}\ldots\delta_p^{k_p}$$

with summation over $j_1, \ldots, j_p, k_1, \ldots, k_p$ subject to $j_1 + \ldots + j_p + k_1 + \ldots + k_p = p, j_e, k_e = 0$ or 1 $(l = 1, \ldots, p), j_e = j_{e+1\,(\mathrm{mod}\,p)} = k_{e+1\,(\mathrm{mod}\,p)} = 0$ if $k_e = 1$ $(l = 1, \ldots, p)$.

This coefficient is invariant with respect to a cyclic permutation of α_i and δ_i $(i = 1, \ldots, p)$. This has the consequence that the continuants of the transpose of A about the secondary diagonal are calculated with the same roots q_1^p and q_2^p.

In what follows the case $q_1^p \neq q_2^p$ is considered. The roots q_1^p and q_2^p determine $2p$ quantities q_k $(k = 1, \ldots, 2p)$. For every coefficients λ_{ik} $(i = 1, \ldots, p)$ are defined by Eq. (4.8). With these quantities q_k and λ_{ik} (Eq. 4.1) yields the general solution of the difference equation

$$u_{mp+i} = \sum_{k=1}^{2p} C_k\lambda_{i+1,k}q_k^{mp+i} = \frac{1}{u_{p-1}}\sum_{k=1}^{p}\left[f_{i+1}(q_1^p)C_kq_k^{mp-1} + f_{i+1}(q_2^p)C_{p+k}q_{p+k}^{mp-1}\right] =$$

$$= f_{i+1}(q_1^p)q_1^{mp}\frac{1}{u_{p-1}}\sum_{k=1}^{p}C_kq_k^{-1} + f_{i+1}(q_2^p)q_2^{mp}\frac{1}{u_{p-1}}\sum_{k=p+1}^{2p}C_kq_k^{-1}$$

$$(m = 0, 1, \ldots;\ i = 0, \ldots, p-1)$$

The factors behind q_1^{mp} and q_2^{mp} are independent of m and i. They are abbreviated to A_1 and A_2. For $f_{i+1}(q^p)$ Eq. (4.9) is substituted. This results in the expression

$$u_{mp+i} = A_1\left(u_iq_1^p + \hat{v}_{p-i-2}\prod_{j=0}^{i}\delta_j\right)q_1^{mp} + A_2\left(u_iq_2^p + \hat{v}_{p-i-2}\prod_{j=0}^{i}\delta_j\right)q_2^{mp}$$

$$(m = 0, 1, \ldots;\ i = 0, \ldots, p-1)$$

The constants A_1 and A_2 are determined from the initial conditions (2.3). The result is $A_1 = -A_2 = 1/(q_1^p - q_2^p)$. With this, the final formula for the continuants is obtained

$$u_{mp+i} = \frac{u_i(q_1^{(m+1)p} - q_2^{(m+1)p}) + (q_1^{mp} - q_2^{mp})\hat{v}_{p-i-2}\prod_{j=0}^{i}\delta_j}{q_1^p - q_2^p} \tag{4.11}$$

$$(m = 0, 1, \ldots;\ i = 0, \ldots, p-1)$$

The derivation of Eqs (4.10) and (4.11) was based on the nonsingularity of $\hat{A}$ $(u_{p-1} \neq 0)$. It can be shown, however, that in the singular case $u_{p-1} = 0$ Eq. (4.10) has the roots $q_1^p = u_p$ and $q_2^p = -\delta_p\hat{v}_{p-2}$ and that with these roots (4.11) is identical with (4.3). Furthermore, Eqs (4.10) and (4.11) reduce to the forms of Eqs (3.2) and (3.7), respectively, if $p = 1$ is substituted. Thus, Eqs (4.10) and (4.11) are valid for arbitrary $p \geqslant 1$ if $q_1^p \neq q_2^p$.

In what follows the solution for continuants is developed for the case $q_1^p = q_2^p$. Only the case $u_{p-1} \neq 0$ need be considered since the case $u_{p-1} = 0$ has been solved already. According to Floquet's theorem the general solution of Eq. (2.2) can be written in the form

$$u_i = [C_1\lambda_{i+1} + C_2(i\lambda_{i+1} + \rho_{i+1})]q^i \quad (i = 0,\dots,n) \tag{4.12}$$

Here, q^p is the double root of Eq. (4.10) and λ_{i+1} $(i = 0, \dots, p-1)$ are the corresponding periodic coefficients determined from Eqs (4.8) and (4.9). The quantities ρ_{i+1} $(i = 0, \dots, p-1)$ are unknown p-periodic coefficients. They are determined by substituting the expression $u_i = (i\lambda_{i+1} + \rho_{i+1})q^i$ into the difference equation (2.2). This results in the equations

$$-\delta_{i-1}\rho_{i-1} + \alpha_i q\rho_i - q^2\rho_{i+1} + i[-\delta_{i-1}\lambda_{i-1} + \alpha_i q\lambda_i - q^2\lambda_{i+1}] =$$

$$= \alpha_i q\lambda_i - 2\delta_{i-1}\lambda_{i-1} \quad (i = 1,\dots,p)$$

According to Eq. (4.2) the expression in square brackets is equal to zero. This enables us to rewrite the right-hand side. After division by q the equations become

$$-\frac{\delta_{i-1}}{q}\rho_{i-1} + \alpha_i\rho_i - q\rho_{i+1} = -\frac{\delta_{i-1}}{q}\lambda_{i-1} + q\lambda_{i+1} \quad (i = 1,\dots,p) \tag{4.13}$$

The coefficient matrix is the same as in (4.2). It is singular since q is an eigenvalue. Its rank is $p-1$ since the $[(p-1)\times(p-1)]$-submatrix $\hat{A}$ shown in Eq. (4.5) has the determinant $u_{p-1} \neq 0$. It follows that only $p-1$ equations are linearly independent and that we are free to choose $\rho_p = 1$.

The constants C_1 and C_2 of Eq. (4.12) are calculated from the initial conditions (2.3). The solution for C_1 is

$$C_1 = \frac{\alpha_1\rho_1 - q(\rho_2 + \lambda_2)}{q[\lambda_2\rho_1 - \lambda_1(\rho_2 + \lambda_2)]}$$

This is equal to zero as is shown by Eq. (4.13) with $i = 1$. From this it follows that $C_2 = 1/\rho_1$ and finally

$$u_i = \frac{1}{\rho_1}(i\lambda_{i+1} + \rho_{i+1})q^i \quad (i = 0,\dots,n) \tag{4.14}$$

Also this formula is valid in the case $p = 1$ in which it is identical with (3.7). Thus, the Eqs (4.10), (4.11) and (4.14) represent the complete solution for the continuants of A in the case $p \geqslant 1$. The same equations yield the continuants of the transpose of A about its secondary diagonal. With u_i and υ_i $(i = 0, \dots, n)$ Eq. (2.1), finally, yields the elements of A^{-1}.

Approximations for symmetric matrices with real roots

In many practical applications the matrix A is symmetric and, furthermore, the roots q_1^p and q_2^p of the characteristic quadratic Eq. (4.10) are real. Their product is the free term in this equation. For a symmetric matrix this term is positive. It follows that the roots have the same signs. Let them be such that $q_2^p/q_1^p < 1$. In Eq. (4.11) terms with q_2 in the numerator are negligible for all $m > m_0$ for some sufficiently large m_0. The smaller q_2^p/q_1^p the smaller is m_0. The approximation is

$$u_{mp+i} \approx \frac{1}{q_1^p - q_2^p}(u_i q_1^p + \hat{\upsilon}_{p-i-2}\prod_{j=0}^{i}\delta_j)q_1^{mp} \quad (m > m_0;\ i = 0,\dots,p-1)$$

whence it follows that

$$u_{j-p}/u_j \approx q_1^{-p} (j - p > m_0 p) \tag{4.15}$$

From Eq. (2.1) it then follows that

$$\frac{(A^{-1})_{i,j-p}}{(A^{-1})_{ij}} = (-1)^p \frac{u_{j-1-p}}{u_{j-1}}\prod_{k=1}^{p}\beta_k \ (i \geqslant j) \tag{4.16}$$

With Eq. (4.15) this yields the approximation

$$(A^{-1})_{i,j-p}/(A^{-1})_{ij} \approx \hat{q} \ (i \geqslant j; j - p > m_0 p + 1) \tag{4.17}$$

where $\hat{q}$ is the constant

$$\hat{q} = (-1)^p q_1^{-p} \prod_{k=1}^{p} \beta_k = (-1)^p \operatorname{sign}\left(q_1^p \prod_{k=1}^{p} \beta_k\right) \sqrt{\frac{q_2^p}{q_1^p}} \quad (|\hat{q}| < 1) \tag{4.18}$$

Here, the fact was used again that q_1^p/q_2^p equals the free term of Eq. (4.10). Replacing in Eq. (4.17) p by mp one finally obtains the approximation

$$(A^{-1})_{i,j-mp} \approx \hat{q}^m (A^{-1})_{i,j} \quad (i \geqslant j;\ j - mp > m_0 p + 1) \tag{4.19}$$

The same reasoning yields

$$(A^{-1})_{i,j+mp} \approx \hat{q}^m (A^{-1})_{i,j} \quad (i \leqslant j;\ j + mp < n - m_0 p) \tag{4.20}$$

These approximations are geometric progressions multiplied by p-periodic functions which are represented by the p elements $(A^{-1})_{i,j-k}$ and $(A^{-1})_{i,j+k}$, respectively, $(k = 0, \ldots, p-1)$.

5. MATRICES WITH SUBMATRICES OF DIFFERENT PERIODS

In this section continuants of a tridiagonal matrix A are determined which has tridiagonal submatrices of different periods. It suffices to solve the following problem. For some ν ($\nu > 1$) the continuants $u_1, \ldots, u_\nu$ have been calculated from given formulas and starting with $\alpha_\nu, \beta_\nu, \gamma_\nu$ the matrix A is p-periodic with $p \geqslant 1$. It is required to obtain an explicit expression for the continuants u_i ($i > \nu$) satisfying the difference equation

$$u_i = \alpha_i u_{i-1} - \delta_{i-1} u_{i-2} \quad (i = \nu + 2, \ldots) \tag{5.1}$$

and given initial conditions $u_{\nu-1}$ and u_ν.

Let U_i and V_i be the linearly independent solutions of Eq. (5.1) with the initial conditions $U_{\nu-1} = 1$, $U_\nu = 0$ and $V_{\nu-1} = 0$, $V_\nu = 1$, respectively. Then

$$u_i = u_{\nu-1} U_i + u_\nu V_i \quad (i = \nu - 1, \ldots) \tag{5.2}$$

is the solution satisfying the initial conditions $u_{\nu-1}$ and u_ν. From Eq. (5.1) and from the initial conditions it follows that

$$U_{\nu+1} = -\delta_\nu, \quad U_{\nu+2} = -\delta_\nu \alpha_{\nu+2}, \quad U_{\nu+3} = -\delta_\nu (\alpha_{\nu+3}\alpha_{\nu+2} - \delta_{\nu+2}),$$

$$V_{\nu+1} = \alpha_{\nu+1}, \quad V_{\nu+2} = \alpha_{\nu+2}\alpha_{\nu+1} - \delta_{\nu+1}, \ldots$$

The general formulas are

$$U_{\nu+i} = -\delta_\nu u_{i-1}^{**}, \quad V_{\nu+i} = u_{i-1}^{*} \quad (i = 1, 2, \ldots) \tag{5.3}$$

Here, u_j^* is the jth continuant of the matrix A^* which remains after deleting from A the rows and columns $1, \ldots, \nu$ and u_j^{**} is the jth continuant of the matrix A^{**} which remains after deleting from A the rows and columns $1, \ldots, \nu + 1$. The matrices A^* and A^{**} are p-periodic and both have the same characteristic roots q_1^p and q_2^p. Their continuants are given by Eq. (4.3) or (4.11) or Eq. (4.14) depending on which of the three cases applies. Equations (5.3) and (5.2) yield the desired explicit formula

$$u_{\nu+i} = u_\nu u_i^* - u_{\nu-1} \delta_\nu u_{i-1}^{**} \quad (i = 1, 2, \ldots) \tag{5.4}$$

6. APPLICATIONS TO MECHANICAL SYSTEMS

Three-moment equations for multiply supported beams

Figure 1(a) shows a continuous beam on supports $0, \ldots, n+1$ and with spans $1, \ldots, n+1$ ($n > 2$). The system is n times statically indeterminate. Figure 1(b) shows an associated statically determinate system with revolute joints at the supports $1, \ldots, n$ and with unknown moments $M_1, \ldots, M_n$. The two systems are equivalent if the moments have such values that the derivative dw/dx of the deflection $w(x)$ is continuous at the supports. This condition is expressed by the so-called three-moment equations

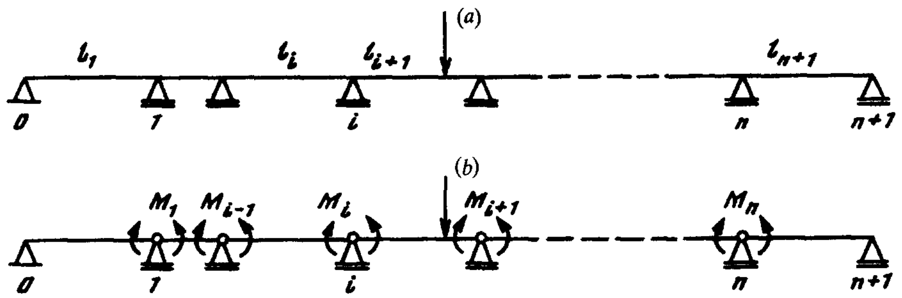

Fig. 1.

$$\frac{l_i}{I_i} M_{i-1} + 2\left(\frac{l_i}{I_i} + \frac{l_{i+1}}{I_{i+1}}\right) M_i + \frac{l_{i+1}}{I_{i+1}} M_{i+1} = Q_{i,i+1} \tag{6.1}$$

$$(i = 1,\ldots,n;\ M_0 = M_{n+1} = 0)$$

The quantity l_i is the length of span i, I_i is the moment of inertia of the cross section in span i and $Q_{i,i+1}$ is a term representing the loads on spans i and $i+1$.

In the special case of identical spans with $l_i/I_i \equiv l/I$ the equations read

$$M_{i-1} + 4M_i + M_{i+1} = Q_{i,i+1} I/l \quad (i = 1,\ldots,n;\ M_0 = M_{n+1} = 0)$$

The coefficient matrix is a symmetric Toeplitz matrix. The quadratic equation (3.2) has the real roots $q_{1,2} = 2 \pm \sqrt{3}$. The elements of the inverse matrix are given by the first line in Eq. (3.14) with $\delta = \beta = 1$

$$(A^{-1})_{ij} = (A^{-1})_{ji} = (-\sqrt{r})^{i-j} \frac{\sqrt{r}}{1-r} \frac{(1-r^j)(1-r^{n+1-i})}{1-r^{n+1}} \quad (i \geqslant j)$$

$$r = (2-\sqrt{3})^2 \approx 0.268^2$$

All the matrix elements except those in rows 1 and n and in columns 1 and n have the good approximation

$$(A^{-1})_{ij} = (A^{-1})_{ji} \approx 0,289(-0,268)^{i-j} \quad (i \geqslant j;\ i,j \neq 1,n)$$

The matrix elements $(1, n)$ and $(n, 1)$ have the correction factor $(1-r)^2 \approx 0.86$ and all other elements of rows 1 and n and of columns 1 and n have the correction factor $(1-r) \approx 0.93$.

As another example, a beam with constant cross section and with three periodically repeated span lengths $l_1 = l$, $l_2 = al$ and $l_3 = bl$ is considered. After multiplication of Eq. (6.1) by I/l the coefficient matrix is symmetric and periodic with period $p = 3$ and with parameters

$$\alpha_1 = 2(1+a),\ \alpha_2 = 2(a+b),\ \alpha_3 = 2(b+1);\quad \beta_1 = a,\ \beta_2 = b,\ \beta_3 = 1$$

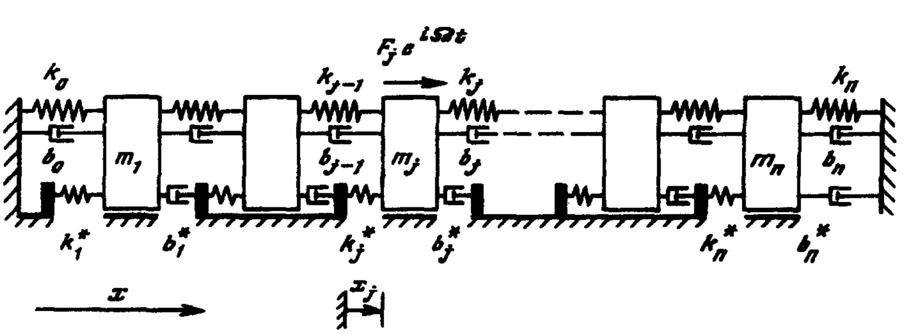

Fig. 2.

The characteristic equation (4.10) is

$$q^6 - 2[3(a+b)(1+a+b+ab)+2ab]q^3 + a^2b^2 = 0$$

Its roots q_1^3 and q_2^3 are real, positive and different from one another. This follows from the fact that their sum, their product as well as the discriminant of the characteristic equation is positive. The continuants are calculated from Eq. (4.11) with

$$u_i = \begin{cases} 1 & (i=0) \\ \alpha_1 & (i=1) \\ \alpha_1\alpha_2 - \beta_1^2 & (i=2) \end{cases}, \quad \vartheta_{p-i-2}\prod_{j=0}^{i}\delta_j = \begin{cases} \alpha_2\beta_3^2 & (i=0) \\ \beta_1^2\beta_3^2 & (i=1) \\ 0 & (i=2) \end{cases}$$

Forced vibrations of chains of bodies

Figure 2 shows a chain of n bodies with masses $m_1, \ldots, m_n$ which are constrained to move along a horizontal support. The bodies and the support are interconnected by two groups of springs and by two groups of dampers. One group of springs has stiffnesses $k_0, \ldots, k_n$ and the other group has stiffnesses $k_1^*, \ldots, k_n^*$. The damping constants are $b_0, \ldots, b_n$ and $b_1^*, \ldots, b_n^*$, respectively. When the system is in equilibrium without external forces the springs may be pre-stressed. Let x_j be the horizontal displacement of body j from this equilibrium position. The bodies are subject to in-phase harmonic excitation forces $F_je^{i\Omega t}$ with a single excitation frequency Ω and with arbitrary real amplitudes F_j ($j = 1, \ldots, n$). The steady response is $x_j(t) = X_je^{i\Omega t}$ ($j = 1, \ldots, n$) where X_j is the complex amplitude of body j. It is required to calculate these amplitudes.

Let m_0 and k_0 be a reference mass and a reference spring constant, respectively. With them, we define

$$\omega_0^2 = \frac{k_0}{m_0}, \quad \tau = \omega_0 t, \quad \mu_j = \frac{m_j}{m_0}, \quad c_j = \frac{k_j}{k_0}, \quad c_j^* = \frac{k_j^*}{k_0}$$

$$2D_j = \frac{b_j}{\sqrt{m_0k_0}}, \quad 2D_j^* = \frac{b_j^*}{\sqrt{m_0k_0}}, \quad \eta = \frac{\Omega}{\omega_0}$$

This leads to the normalized equations of motion (the prime denotes $d/d\tau$)

$$\mu_jx_j'' - 2D_{j-1}x_{j-1}' + 2(D_j^* + D_{j-1} + D_j)x_j' - 2D_jx_{j+1}' -$$

$$-c_{j-1}x_{j-1} + (c_j^* + c_{j-1} + c_j)x_j - c_jx_{j+1} = (1/k_0)F_je^{i\eta\tau}$$

$$(j = 1, \ldots, n; \;\; x_0 = x_{n+1} \equiv 0)$$

Substitution of $x_j(\tau) = X_je^{i\eta\tau}$ ($j = 1, \ldots, n$) produces a system of linear equations for the desired stationary amplitudes. Its matrix form is $AX = (1/k_0)F$ with a symmetric, tridiagonal coefficient matrix A having the complex parameters

$$\alpha_j = -\mu_j\eta^2 + (c_j^* + c_{j-1} + c_{j+1}) + i2\eta(D_j^* + D_{j-1} + D_{j+1}) \;\; (j = 1, \ldots, n)$$

$$\beta_j = -c_j - i2D_j\eta, \quad \delta_j = \beta_j^2 \;\; (j = 1, \ldots, n-1)$$

They are p-periodic ($p \geqslant 1$) if the masses, stiffnesses and/or damper constants are p-periodic. Explicit results for the inverse elements are given only for an undamped system with identical masses ($m_j \equiv m$), with $k_0 = k_n = 0$ and with two groups of identical stiffnesses: $k_j \equiv k$ ($j = 2, \ldots, n-1$), $k_j^* \equiv k^*$ ($j = 1, \ldots, n$). Choosing $m_0 = m$ and $k_0 = k^*$ we get $\mu_j \equiv 1$, $c_j \equiv k/k^* = c$, $c_j^* \equiv 1$ ($j = 1, \ldots, n$). With this the matrix A is a symmetric Toeplitz matrix with perturbations. Its parameters are $\alpha = 1 - \eta^2 + 2c$, $\beta = \gamma = -c$ and $\alpha_1 = \alpha + \beta$. This is the case of Eq. (3.15) with $\sigma_1 = 1$. The parameters r and φ defined by Eqs (3.9) and (3.12) are

$$r(\eta, c) = \frac{1-\zeta_+}{1+\zeta_+}, \quad \varphi(\eta, c) = \operatorname{arctg}\zeta_- \quad (0 < \varphi \leqslant \pi/2) \tag{6.2}$$

$$\zeta_\pm = \sqrt{\pm(1-\eta^2)(1-\eta^2+4c)}/|1-\eta^2+2c|$$

The inverse matrix is given by Eq. (3.22)

$$(A^{-1})_{ij} = (A^{-1})_{ji} = \begin{cases} \dfrac{1}{c}(\sqrt{r})^{i-j+1}\dfrac{(1+r^{j-\frac{1}{2}})(1+r^{n-i+\frac{1}{2}})}{(1-r)(1-r^n)} & (\eta^2 < 1) \\ A \text{ is singular} & (\eta^2 = 1) \\ \dfrac{1}{c}\dfrac{\cos[(j-\frac{1}{2})\varphi]\cos[(n-i+\frac{1}{2})\varphi]}{\sin\varphi\sin n\varphi} & (1<\eta^2<1+2c) \\ \dfrac{1}{c}(-1)^{i-j+1}\dfrac{\sin[(j-\frac{1}{2})\varphi]\sin[(n-i+\frac{1}{2})\varphi]}{\sin\varphi\sin n\varphi} & (1+2c<\eta^2<1+4c) \\ \dfrac{-1}{c}\dfrac{(2j-1)[2(n-i)+1]}{4n} & (\eta^2 = 1+4c) \\ \dfrac{1}{c}(-\sqrt{r})^{i-j+1}\dfrac{(1-r^{j-\frac{1}{2}})(1-r^{n-i+\frac{1}{2}})}{(1-r)(1-r^n)} & (\eta^2 > 1+4c) \end{cases}$$

$$(j=1,\ldots,n,\ i \geqslant j)$$

The system has n eigenfrequencies given by $\eta = 1$ and by those values of η for which $\sin n\varphi(\eta, c) = 0$. The solution is of interest also in the static case, i.e. for $\eta = 0$. In this case, the body masses and dampers do not play any role. In the equations they are set equal to zero. The springs can be arbitrary elastic structures. Eq. (6.2) becomes

$$r = \left(\frac{\sqrt{1+4c}-1}{\sqrt{1+4c}+1}\right)^2$$

whence it follows that $1/c = (1-\sqrt{r})^2/\sqrt{r}$. With these expressions the inverse elements are functions of r, i and j only.

Another elastic system under static loading is shown in Fig. 3. It consists of identical elastic bodies labelled $1, 3, \ldots, n-1$ and of identical springs (of stiffness k) with attachment points $0, 1, \ldots, n+1$. External forces F_i applied to the points $i = 1, 2, \ldots, n$ cause static displacements x_i of these points. These displacements are to be calculated. The bodies can have other forms than those shown in the figure. For an individual body the stiffness matrix K is given. It relates the displacements x_i and x_{i+1} to the forces R_i and R_{i+1} acting at the same points

$$\begin{Vmatrix} K_{11} K_{12} \\ K_{12} K_{22} \end{Vmatrix} \begin{Vmatrix} x_i \\ x_{i+1} \end{Vmatrix} = \begin{Vmatrix} R_i \\ R_{i+1} \end{Vmatrix} \tag{6.3}$$

The element K_{12} is negative, zero or positive depending on the shape of the body. In what follows $K_{12} \neq 0$ is assumed since otherwise the bodies are uncoupled.

The forces acting on body i ($i = 1, 3, \ldots, n-1$) are $R_i = F_i - k(x_i - x_{i-1})$ at point i and $R_{i+1} = F_{i+1} + k(x_{i+2} - x_{i+1})$ at point $i+1$. This is substituted into Eq. (6.3)

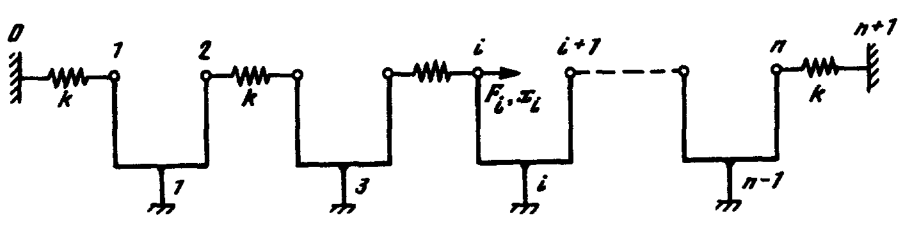

Fig. 3.

$$\begin{Vmatrix} -k & k+K_{11} & K_{12} & 0 \\ 0 & K_{12} & k+K_{22} & -k \end{Vmatrix} \begin{Vmatrix} x_{i-1} \\ x_i \\ x_{i+1} \\ x_{i+2} \end{Vmatrix} = \begin{Vmatrix} F_i \\ F_{i+1} \end{Vmatrix}$$

$$(i = 1, 3, \ldots, n-1; \quad x_0 = x_{n+1} = 0)$$

After division by k all pairs of equations are combined in the matrix equation $Ax = (1/k)F$. The symmetric coefficient matrix A has period $p = 2$ and the parameters $\alpha_1 = 1 + K_{11}/k > 0$, $\alpha_2 = 1 + K_{22}/k > 0$, $\beta_1 = K_{12}/k$ (< 0 or > 0) and $\beta_2 = -1$. The characteristic Eq. (4.4) is

$$q^4 - [(K_{11} + K_{22})/k + (K_{11}K_{22} - K_{12}^2)/k^2]q^2 + (K_{12}/k)^2 = 0$$

Its roots q_1^2 and q_2^2 are real, positive and different from one another. Hence, the approximations (4.19) and (4.12) are applicable. Equation (4.11) yields the positive continuants

$$u_{2m+i} = \begin{cases} [(1+q_1^2)q_1^{2m} - (1+q_2^2)q_2^{2m}]/(q_1^2 - q_2^2) & (i = 0) \\ \alpha_1[q_1^{2(m+1)} - q_2^{2(m+1)}]/(q_1^2 - q_2^2) & (i = 1) \end{cases} \qquad (m = 0, 1, \ldots)$$

The continuants υ_{2m+i} of the transpose about the secondary diagonal are the same expressions with α_2 instead of α_1. With the quantities Eq. (2.1) yields the elements of the inverse matrix. If $K_{12} < 0$ then all elements of A^{-1} are positive. If $K_{12} > 0$ then all diagonal elements of A^{-1} are positive, and in each row and in each column the sign distribution is [... + + − − + + − − ...]. These sign distributions were to be expected by simple physical reasoning.

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